

Crossed Modules of Algebras as Ideal Maps

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Abstract

In this work, we explore the close relationship between an ideal map structure $S \rightarrow \text{End}(R)$ on a homomorphism of commutative k -algebras $R \rightarrow S$ and an ideal simplicial algebra structure on the associated bar construction $\text{Bar}(S, R)$.

Introduction

In this paper, we consider the equivalence between the category of crossed modules of algebras (cf. [11]) and the category of simplicial commutative algebras with Moore complex of length 1 given in [2]. The main aim of this note is to associate an explicit ideal simplicial algebra structure on the bar construction given a crossed module of algebras. We observed that a crossed module structure $(S \rightarrow \text{End}(R))$ or an ideal map structure on a homomorphism of algebras $\eta : R \rightarrow S$, yields directly a simplicial algebra structure on the usual bar construction namely on the simplicial k -module

$$\text{Bar}(S, R) = (S \times R^k)_{k \geq 0}.$$

Thus $\text{Bar}(S, R)$ is isomorphic, as a simplicial k -module, to a simplicial algebra which is compatible with the action of R on the bar construction. Moreover this process is reversible. Therefore we can summarize the result as follows: Given an algebra homomorphism $\eta : R \rightarrow S$, a crossed module structure or an ideal map structure on the homomorphism η gives an ideal simplicial algebra structure on the simplicial k -module $\text{Bar}(S, R)$, and conversely, any ideal simplicial algebra structure on the simplicial k -module $\text{Bar}(S, R)$ determines a crossed module structure on the homomorphism η . These two explicit associations are mutual inverses. In the last section, we are explaining how to give an extension of this result to Ellis's (crossed) squares of k -algebras (cf. [6]). In section 5, considering a *crossed ideal structure* over the map $\alpha : \eta_1 \rightarrow \eta_2$ between crossed modules η_1 and η_2 , we proved that a *crossed ideal map* preserves the *crossed ideals* in the category of crossed modules of commutative k -algebras.

These constructions in the category of groups can be found in [8]. In fact, the results and general methods given in this work are inspired by those proved for the corresponding case of groups using homotopy normal maps in [8]. For further work about homotopy normal maps, see [7] and [12] and for the free normal closure of a homotopy normal map see [9].

1 Simplicial sets and simplicial algebras

Let k be a fixed commutative ring with identity. By a k -algebra, we mean a unitary k -bimodule C endowed with a k -bilinear associative multiplication $C \times C \rightarrow C$, $(c, c') \mapsto cc'$. The algebra C will as usual be called commutative if $cc' = c'c$ for all $c, c' \in C$. In this work, all algebras will be commutative and will be over the same fixed commutative ring k . We will denote the category of all algebras over the commutative ring k by \mathbf{Alg} .

A simplicial set E consists of a family of sets $\{E_n\}$ together with face and degeneracy maps $d_i = d_i^n : E_n \rightarrow E_{n-1}$, $0 \leq i \leq n$, ($n \neq 0$) and $s_i = s_i^n : E_n \rightarrow E_{n+1}$, $0 \leq i \leq n$, satisfying the usual simplicial identities given in André [1] or Illusie [10] for example. It can be completely described as a functor $E : \Delta^{op} \rightarrow \mathbf{Sets}$ where Δ is the category of finite ordinals $[n] = \{0 < 1 < \dots < n\}$ and non-decreasing maps.

We say that the simplicial set E is a simplicial k -module (or k -algebra) if E_k is a k -module (or a k -algebra), for all k and the face and degeneracy maps are homomorphisms of k -modules (or k -algebras). Thus, a simplicial algebra can be defined as a functor from the opposite category Δ^{op} to \mathbf{Alg} .

1.1 The simplicial k -module $Bar(X, R)$

In this section, we use the usual bar construction of a simplicial k -module by using the action of a k -algebra on a k -module. First, we define this action.

Let R be a k -algebra and X be a k -module. The action of R on the k -module X is defined by the function $X \times R \rightarrow X$, $x : r \mapsto x^r$ (where $r \in R, x \in X$) satisfying the following conditions:

1. $(x)^{(r_1+r_2)} = (x^{r_1})^{r_2}$
2. $x^{0_R} = x$
3. $(x_1 + x_2)^{r_1+r_2} = (x_1)^{r_1} + (x_2)^{r_2}$
4. $k(x)^r = (kx)^{kr}$

for all $r, r_1, r_2 \in R, x, x_1, x_2 \in X, k \in k$.

Let R be a k -algebra acting on the k -module X . Then we obtain a k -module $B_n = X \times R^n$ together with the operations

$$(x, r_1, r_2, \dots, r_n) \oplus (x', r'_1, r'_2, \dots, r'_n) = (x + x', r_1 + r'_1, \dots, r_n + r'_n)$$

for $x, x' \in X$ and $r_i, r'_i \in R_i$ and

$$k(x, r_1, r_2, \dots, r_n) = (kx, kr_1, kr_2, \dots, kr_n)$$

for $k \in k$.

The bar construction

$$B := \text{Bar}(X, R)$$

is the simplicial \mathbf{k} -module consisting of the following data.

1. for each integer $n \geq 0$, a \mathbf{k} -module B_n defined by $B_0 = X$ for $n = 0$, and $B_n = X \times R^n$ is the \mathbf{k} -module as described above for $n \geq 1$, together with
2. the face \mathbf{k} -module homomorphisms $d_i^n : d_i : B_n \rightarrow B_{n-1}$ for all $n \geq 1$ and $0 \leq i \leq n$ defined by:

$$(i) \ d_0 : (x, r_1, r_2, \dots, r_n) \mapsto (x^{r_1}, r_2, \dots, r_n)$$

$$(ii) \ d_i : (x, r_1, r_2, \dots, r_i, r_{i+1}, \dots, r_n) \mapsto (x, r_1, r_2, \dots, r_i + r_{i+1}, \dots, r_n) \text{ for } 1 \leq i < n,$$

$$(iii) \ d_n : (x, r_1, r_2, \dots, r_n) \mapsto (x, r_1, r_2, \dots, r_{n-1}),$$

3. and together with degeneracy \mathbf{k} -module homomorphisms; $s_i : B_n \rightarrow B_{n+1}$ defined by

$$s_i : (x, r_1, r_2, \dots, r_n) \mapsto (x, r_1, r_2, \dots, r_i, 0, r_{i+1}, \dots, r_n)$$

for all $n \geq 0$ and $0 \leq i \leq n$.

In this construction we show briefly that d_0 is a \mathbf{k} -module homomorphism from B_n to B_{n-1} . For $u = (x, r_1, r_2, \dots, r_n), v = (x', r'_1, r'_2, \dots, r'_n) \in B_n$ and $k \in \mathbf{k}$, we obtain

$$\begin{aligned} d_0(u \oplus v) &= d_0(x + x', r_1 + r'_1, \dots, r_n + r'_n) \\ &= ((x + x')^{r_1 + r'_1}, r_2 + r'_2, \dots, r_n + r'_n) \\ &= (x^{r_1} + (x')^{r'_1}, r_2 + r'_2, \dots, r_n + r'_n) \\ &= (x^{r_1}, r_2, \dots, r_n) \oplus ((x')^{r'_1}, r'_2, \dots, r'_n) \\ &= d_0(u) \oplus d_0(v) \end{aligned}$$

and

$$\begin{aligned} d_0(ku) &= d_0(kx, kr_1, \dots, kr_n) \\ &= ((kx)^{kr_1}, kr_2, \dots, kr_n) \\ &= (k(x^{r_1}), kr_2, \dots, kr_n) \\ &= k(x^{r_1}, r_2, \dots, r_n) \\ &= kd_0(u). \end{aligned}$$

1.2 An ideal simplicial algebra structure on $\text{Bar}(S, R)$

Suppose now that $X = S$ is a \mathbf{k} -algebra and the \mathbf{k} -algebra R acts on the underlying \mathbf{k} -module S of the \mathbf{k} -algebra S via a homomorphism $\eta : R \rightarrow S$, i.e. the action is

$$r : s \rightarrow s^r = s + \eta(r)$$

for all $r \in R$ and $s \in S$. In this case we obtain

1. $s^{(r_1+r_2)} = s + \eta(r_1 + r_2) = (s + \eta(r_1)) + \eta(r_2) = (s^{r_1})^{r_2}$
2. $s^{0_R} = s + \eta(0_R) = s + 0_S = s$
3. $(s_1 + s_2)^{(r_1+r_2)} = (s_1 + s_2) + \eta(r_1 + r_2) = s_1 + \eta(r_1) + s_2 + \eta(r_2) = (s_1)^{r_1} + (s_2)^{r_2}$
4. $k(s)^r = k(s + \eta(r)) = ks + \eta(kr) = (ks)^{kr}$

for all $s, s_1, s_2 \in S$ and $r, r_1, r_2 \in R$ and $k \in \mathbf{k}$. We denote the resulting simplicial \mathbf{k} -module by $\text{Bar}(S, R)$ suppressing the map η .

Definition 1.1 Let $B := \text{Bar}(S, R)$. By an ideal simplicial algebra structure on B we mean the following

- (i) $B_0 = S$ is the \mathbf{k} -algebra S ,
- (ii) $B_k := S \times R^k$ for $k \geq 1$, is endowed with a \mathbf{k} -algebra structure for all $k \geq 1$, we denote the multiplication by

$$(s, r_1, \dots, r_k) * (s', r'_1, \dots, r'_k).$$

- (iii) the face d_i^k and the degeneracy s_j^k \mathbf{k} -module homomorphisms are \mathbf{k} -algebra homomorphisms.

(iv)

$$(s, 0, \dots, 0) * (s', r'_1, \dots, r'_k) = (ss', 0, \dots, 0)$$

for all $s, s' \in S$ and $(s', r'_1, \dots, r'_k) \in B_k$ where the operations takes place in B_k .

Remark 1.2 By the natural action of S on $\text{Bar}(S, R)$, we mean

$$s : (s', r_1, \dots, r_k) \mapsto s \cdot (s', r_1, \dots, r_k) = (ss', r_1, \dots, r_k)$$

for all $k \geq 0$, $(s', r_1, \dots, r_k) \in B_k$ and $s \in S$. When we say that the multiplication in $\text{Bar}(S, R)$ is compatible with the natural action of S , we mean that condition (iv) of the above definition holds.

Notation 1.3 Let $k \geq 1$. We denote

1. $S_k := \{(s, 0_R, 0_R, \dots, 0_R) : s \in S\}$ is a subalgebra of B_k .
2. $R_k := \{(0_S, r_1, r_2, \dots, r_k) : r_i \in R\}$ is an algebra ideal of B_k .

Lemma 1.4 *Suppose that $\text{Bar}(S, R)$ is endowed with an ideal simplicial algebra structure. Let $k \geq 1$. Then S_k is an ideal of B_k which is isomorphic to S , R_k is an ideal of B_k , $B_k = S_k + R_k$ and $S_k \cap R_k = \{0\}$.*

Proof: S_k is the image of S_{k-1} under s_{k-1} , so by induction it is a subalgebra of B_k and s_{k-1} is injective, it is isomorphic to S . Also, R_k is the kernel of $d_k \circ d_{k-1} \circ \cdots \circ d_1$, so it is an ideal of B_k . Clearly $S_k \cap R_k = \{0\}$ and $B_k = S_k + R_k$. \square

1.3 Crossed modules, ideal maps and ideal structures

Crossed modules of groups were initially defined by Whitehead in [15]. The algebra analogue has been studied by Porter in [11].

A crossed module of algebras consists of an algebra homomorphism $\eta : R \rightarrow S$ which we call here an ideal map (see Remark 1.5) together with a homomorphism $l : S \rightarrow \text{End}(R)$ which we call here an ideal structure (or a crossed module structure) on η such that when denoting by $s \cdot r$ the image of $r \in R$ under l_s for $s \in S$ which is satisfying the conditions below (for all $k \in \mathbf{k}$, $r, r' \in R$ and $s, s' \in S$)

1. $k(s \cdot r) = (ks) \cdot r = s \cdot (kr)$
2. $s \cdot (r + r') = s \cdot r + s \cdot r'$
3. $(s + s') \cdot r = s \cdot r + s' \cdot r$
4. $s \cdot (rr') = (s \cdot r)r' = r(s \cdot r')$
5. $(ss') \cdot r = s \cdot (s' \cdot r)$

and the following two requirements are satisfied:

- (CM1) $\eta(l_s(r)) = s\eta(r)$ for all $s \in S$ and $r \in R$.
- (CM2) $l_{\eta(r)}(r') = rr'$ for all $r, r' \in R$.

Remark 1.5 Let S and R be algebras and let $\eta : R \rightarrow S$ be an algebra homomorphism. If $l_s : S \rightarrow \text{End}(R)$; $s \in S$ is a crossed module structure on the homomorphism $\eta : R \rightarrow S$, then $\text{Im}(\eta)$ becomes an ideal of S . Indeed, for all $s \in S$ and $s' \in \text{Im}(\eta)$ with $s' = \eta(r)$; $r \in R$, we obtain from (CM1),

$$ss' = s\eta(r) = \eta(l_s(r)) \in \text{Im}(\eta).$$

Thus $\text{Im}(\eta)$ is an ideal of S . Conversely, if I is an ideal of the algebra S , then the inclusion map $I \rightarrow S$ is a crossed module with the natural action of S on I . Further $\ker \eta$ is an ideal in R and a module over S . The ideal $\text{Im}(\eta)$ of S acts trivially on $\ker \eta$, hence $\ker \eta$ inherits an action of $S/\text{Im}(\eta)$ to become an $S/\text{Im}(\eta)$ -module.

Now let S be an algebra and R be subalgebra of S . Let $\eta : R \rightarrow S$ be the inclusion map and let S/R be the set of cosets of R in S . Then there is a natural action of S on the set S/R via left multiplication and it is easy to check that the following statements are equivalent.

- (i) R is an ideal of S .
- (ii) There exists a crossed module structure on the inclusion map $\eta : R \rightarrow S$.
- (iii) There exists an algebra structure on S/R with the action of S on S/R given by

$$s \cdot (s' + R) = ss' + R$$

for all $s, s' \in S$.

2 From an ideal simplicial algebra structure on $Bar(S, R)$ to an ideal structure on the map $\eta : R \rightarrow S$

In this section R and S are algebras and $\eta : R \rightarrow S$ is an algebra homomorphism. The purpose of this section is to prove that we can recover the crossed module structure (or an ideal structure) on a homomorphism between k -algebras, from an ideal simplicial algebra structure on the associated bar construction.

Lemma 2.1 *Suppose that $Bar(S, R)$ is endowed with an ideal simplicial algebra structure. Then*

1. $(0, r) \oplus (0, r') = (0, r + r')$ and $(0, r) * (0, r') = (0, rr')$ for all $r, r' \in R$ where the operations take place in R_1 .
2. The map $l : S \rightarrow End(R)$ defined by

$$l_s : (r) \mapsto s \cdot r$$

gives an ideal structure (or a crossed module structure) on η , where

$$(0, s \cdot r) = (s, 0) * (0, r).$$

Lemma 2.2 *Let $k \geq 0$ and $r, r' \in R$. Then*

- (i) *The zero element of B_k is $(0_S, 0_R, \dots, 0_R)$,*
- (ii) $(0, -r, r) \oplus (0, 0, r') = (0, -r, r + r')$,
- (iii) $(-\eta(r), r) \oplus (0, r') = (-\eta(r), r + r')$,
- (iv) $(0, r) * (0, r') = (0, rr')$.

Proof:

(i) By definition, the zero element of $B_0 = S$ is the zero element 0_S of S . Then by induction since $s_0 : B_k \rightarrow B_{k+1}$ is an algebra homomorphism, for all $k \geq 0$, part (i) follows.

For example $s_0(0_S) = (0_S, 0_R)$ is the zero element of B_1 and $s_0(0_S, 0_R) = (0_S, 0_R, 0_R)$ is the zero element of B_2 and so on.

(ii) Applying d_2^2 and using (i) we get that

$$(0_S, -r, r) \oplus (0_S, 0_R, r') = (0_S, -r, x).$$

Applying d_1^2 and using (i) again we get that

$$(0_S, r') = (0_S, -r + x)$$

so; $x = r + r'$ and (ii) holds.

(iii) This part follows from (ii) by applying d_0^2 . \square

Notice that $B_1 = S \ltimes R$ is a semidirect product algebra of R by S , that is, the addition and the multiplication in B_1 are given respectively by

$$(s, r) \oplus (s', r') = (s + s', r + r')$$

and

$$(s, r) * (s', r') = (ss', s \cdot r' + s' \cdot r + rr')$$

for all $s, s' \in S$ and $r, r' \in R$.

Lemma 2.3 *The map $\Phi : S \ltimes R \rightarrow S$ defined by $\Phi(s, r) = s + \eta(r)$ is a homomorphism of algebras if and only if η satisfies (CM1) above.*

Proof: For all $(s, r), (s', r') \in S \ltimes R$, we have

$$\begin{aligned} \Phi((s, r) \oplus (s', r')) &= \Phi((s + s', r + r')) \\ &= s + s' + \eta(r + r') \\ &= s + \eta(r) + s' + \eta(r') \\ &= \Phi(s, r) + \Phi(s', r') \end{aligned}$$

and

$$\begin{aligned} \Phi((s, r) * (s', r')) &= \Phi(ss', s \cdot r' + s' \cdot r + rr') \\ &= ss' + \eta(s \cdot r' + s' \cdot r + rr') \\ &= ss' + s\eta(r') + s'\eta(r) + \eta(r)\eta(r') \quad \text{since (CM1)} \\ &= s(s' + \eta(r')) + \eta(r)(s' + \eta(r')) \\ &= (s + \eta(r))(s' + \eta(r')) \\ &= \Phi((s, r))\Phi((s', r')). \end{aligned}$$

\square

Lemma 2.4 Consider the action of R on itself via multiplication and form the semidirect product $R \ltimes R$ with respect to this action. Thus

$$(a, b) \oplus (c, d) = (a + c, b + d)$$

and

$$(a, b) * (c, d) = (ac, ad + bc + bd), a, b, c, d \in R.$$

Then the map $\Phi : R \ltimes R \rightarrow S \ltimes R$ defined by $(a, b) \mapsto (\eta(a), b)$ is a homomorphism if and only if η satisfies (CM2).

Proof: For all $(a, b), (c, d) \in R \ltimes R$, we obtain

$$\begin{aligned} \Phi((a, b) \oplus (c, d)) &= \Phi((a + c, b + d)) \\ &= (\eta(a + c), b + d) \\ &= (\eta(a), b) + (\eta(c), d) \\ &= \Phi(a, b) + \Phi(c, d) \end{aligned}$$

and

$$\begin{aligned} \Phi((a, b))\Phi((c, d)) &= (\eta(a), b)(\eta(c), d) \\ &= (\eta a \eta c, \eta(a) \cdot d + \eta(c) \cdot b + bd) \\ &= (\eta(ac), ad + bc + bd) \quad \text{since (CM2)} \\ &= \Phi(ac, ad + bc + bd) \\ &= \Phi((a, b) * (c, d)). \end{aligned}$$

□

Lemma 2.5 Let $a_i, b_i \in R$. Then

(i)

$$(0_S, a_1, \dots, a_k) * (0_S, b_1, \dots, b_k) = (0_S, a_1 b_1, a_1 b_2 + a_2(b_1 + b_2), \dots, (\sum_{i=1}^{k-1} a_i) b_k + a_k \sum_{i=1}^k b_i).$$

(ii) Let $s \in S$ and $(0_S, a_1, a_2, \dots, a_k) \in R_k$. Then

$$(0_S, a_1, \dots, a_k) * (s, 0_R, \dots, 0_R) = (0_S, s \cdot a_1, s \cdot a_2, \dots, s \cdot a_k).$$

Proof: We prove (i) by induction on k . For $k = 1$, it is easy to see that

$$(0_S, a_1) * (0_S, b_1) = (0_S, a_1 b_1).$$

Then by applying d_k using induction we see that

$$(0_S, a_1, \dots, a_k) * (0_S, b_1, \dots, b_k) = (0_S, a_1 b_1, \dots, (\sum_{i=1}^{k-2} a_i) b_{k-1} + a_{k-1} \sum_{i=1}^{k-1} b_i, x).$$

Applying d_{k-1} using induction once more we get that

$$\begin{aligned}
& (0_S, a_1 b_1, \dots, (\sum_{i=1}^{k-2} a_i) b_{k-1} + a_{k-1} \sum_{i=1}^{k-1} b_i + x) \\
&= (0_S, a_1, \dots, a_{k-1} + a_k) * (0_S, b_1, \dots, b_{k-1} + b_k) \\
&= (0_S, a_1 b_1, \dots, \sum_{i=1}^{k-2} a_i (b_{k-1} + b_k) + (a_{k-1} + a_k) \sum_{i=1}^k b_i) \\
&= (0_S, a_1 b_1, \dots, \sum_{i=1}^{k-2} a_i (b_{k-1}) + \sum_{i=1}^{k-2} a_i (b_k) + a_{k-1} \sum_{i=1}^k b_i + a_k \sum_{i=1}^k b_i) \\
&= (0_S, a_1 b_1, \dots, \sum_{i=1}^{k-2} a_i (b_{k-1}) + a_{k-1} \sum_{i=1}^{k-1} b_i + a_{k-1} b_k + \sum_{i=1}^{k-2} a_i (b_k) + (a_k) \sum_{i=1}^k b_i).
\end{aligned}$$

It follows that

$$x = (\sum_{i=1}^{k-1} a_i) b_k + a_k \sum_{i=1}^k b_i.$$

(ii) By induction on k similarly, we prove Part (ii). For $k = 1$, we have

$$(0_S, a_1) * (s, 0_R) = (0_S, s \cdot a_1).$$

Applying d_k using induction we see that for $k - 1$

$$(0_S, a_1, \dots, a_k) * (s, 0_R, \dots, 0_R) = (0_S, s \cdot a_1, s \cdot a_2, \dots, s \cdot a_{k-1}, x).$$

Then applying d_{k-1} using induction, we get that

$$\begin{aligned}
(0_S, s \cdot a_1, \dots, s \cdot a_{k-1} + x) &= (0_S, a_1, \dots, a_{k-1} + a_k) * (s, 0_R, \dots, 0_R) \\
&= (0_S, s \cdot a_1, \dots, s \cdot (a_{k-1} + a_k))
\end{aligned}$$

and so, $x = s \cdot a_k$. \square

Proposition 2.6 *The homomorphism $l : S \rightarrow \text{End}(R)$ is an ideal structure (or a crossed module structure) on the map $\eta : R \rightarrow S$.*

Proof: Since $B_1 = S \ltimes R$, and since the homomorphism

$$d_0 : S \ltimes R = B_1 \rightarrow B_0 = S$$

is defined by $d_0(s, r) = s^r = s + \eta(r)$, Lemma 2.3 implies that (CM1) holds for the map $\eta : R \rightarrow S$. Notice that by Lemma 2.5 the subalgebra R_2 is isomorphic to $R \ltimes R$. Further, the map d_0 restricted to R_2 is given by $d_0(0_S, a, b) = (\eta(a), b)$ and it is a homomorphism from $R \ltimes R$ to $S \ltimes R$ given by $(a, b) \mapsto (\eta(a), b)$. Hence by Lemma 2.4, (CM2) holds for the map η . \square

Let $(s, a_1, \dots, a_k), (s', b_1, \dots, b_k) \in B_k$. Then from the above results we get

$$(s, a_1, \dots, a_k) * (s', b_1, \dots, b_k) = (ss', s \cdot b_1 + s' \cdot a_1 + a_1 b_1, s \cdot b_2 + s' \cdot a_2 + a_1 b_2 + a_2(b_1 + b_2), \\ \dots, s \cdot b_k + s' \cdot a_k + \sum_{i=1}^{k-1} a_i b_k + a_k \sum_{i=1}^k b_i)$$

and

$$(s, a_1, \dots, a_k) \oplus (s', b_1, \dots, b_k) = (s + s', a_1 + b_1, \dots, a_k + b_k).$$

3 From an ideal structure on $\eta : R \rightarrow S$ to an ideal simplicial algebra structure on $\text{Bar}(S, R)$.

In this section S and R are algebras and $\eta : R \rightarrow S, l : S \rightarrow \text{End}(R)$ are algebra homomorphism. Recall that we denote

$$l_s : r \mapsto l_s(r) = s \cdot r$$

for $s \in S$ and $r \in R$.

We assume that l is an ideal structure or a crossed module structure on η . We let $\text{Bar}(S, R)$ denote the bar construction using the action of the k -algebra R on the underlying k -module S of the algebra S via $s \mapsto s + \eta(r)$ for all $s \in S$ and $r \in R$. Our aim in this section is to show that the crossed module structure l leads to an ideal simplicial algebra structure on $\text{Bar}(S, R)$.

We start by defining a multiplication on B_k for all $k \geq 0$. For $k = 0$, $B_0 = S$ and the operations are as in S . Obviously, from simplicial structure $\text{Bar}(S, R)$, for $k \geq 1$, we can denote the addition by

$$(s, a_1, \dots, a_k) \oplus (s', b_1, \dots, b_k) = (s + s', a_1 + b_1, \dots, a_k + b_k).$$

We can define the multiplication by

$$(s, a_1, \dots, a_k) * (s', b_1, \dots, b_k) = (ss', s \cdot b_1 + s' \cdot a_1 + a_1 b_1, s \cdot b_2 + s' \cdot a_2 + a_1 b_2 + a_2(b_1 + b_2), \\ \dots, s \cdot b_k + s' \cdot a_k + \sum_{i=1}^{k-1} a_i b_k + a_k \sum_{i=1}^k b_i)$$

as illustrated above.

Theorem 3.1 *Let $k \geq 0$. Then*

- (i) B_k is an algebra,
- (ii) the k -module homomorphism

$$d_0 : (s, a_1, \dots, a_k) \mapsto (s + \eta(a_1), a_2, \dots, a_k)$$

is a k -algebra homomorphism from B_k to B_{k-1} ,

(iii) the k -module homomorphisms

$$d_i : (s, a_1, \dots, a_k) \mapsto (s, a_1, \dots, a_{i-1} + a_i, \dots, a_k)$$

are k -algebra homomorphisms from B_k to B_{k-1} for all $1 \leq i \leq k-1$,

(iv) the k -module homomorphism

$$d_k : (s, a_1, \dots, a_k) \mapsto (s, a_1, \dots, a_{k-1})$$

is a k -algebra homomorphism from B_k to B_{k-1} ,

(v) the k -module homomorphisms

$$s_i : (s, a_1, \dots, a_k) \mapsto (s, a_1, \dots, a_i, 0, a_{i+1}, \dots, a_k)$$

are k -algebra homomorphisms for all $0 \leq i \leq k$.

Proof: (i) For each $k \geq 1$ define

$$\eta_k : (s, a_1, \dots, a_k) \mapsto s + \eta(a_1 + \dots + a_k)$$

from B_k to S . We prove that B_k is an algebra and that η_k is an algebra homomorphism. For $k = 1$, this is Lemma 2.3. Suppose this holds for $k-1$. Then B_{k-1} acts on R via

$$(s, a_1, \dots, a_{k-1}) : a \mapsto a \cdot (s + \eta(a_1 + \dots + a_{k-1}))$$

for $(s, a_1, \dots, a_{k-1}) \in B_{k-1}$ and $a \in R$. Notice that B_k is just the semi-direct product algebra $B_{k-1} \ltimes R$ with respect to this action, so B_k is an algebra. To show that η_k is an algebra homomorphism we obtain

$$\begin{aligned} & \eta_k((s, a_1, \dots, a_k) * (s', b_1, \dots, b_k)) \\ &= \eta_k(ss', s \cdot b_1 + s' \cdot a_1 + a_1 b_1, s \cdot b_2 + s' \cdot a_2 + a_1 b_2 + a_2(b_1 + b_2), \\ & \quad \dots, s \cdot b_k + s' \cdot a_k + \sum_{i=1}^{k-1} a_i b_k + a_k \sum_{i=1}^k b_i) \\ &= ss' + \eta(s \cdot b_1 + s' \cdot a_1 + a_1 b_1 + s \cdot b_2 + s' \cdot a_2 + a_1 b_2 + a_2(b_1 + b_2) + \\ & \quad \dots + s \cdot b_k + s' \cdot a_k + \sum_{i=1}^{k-1} a_i b_k + a_k \sum_{i=1}^k b_i) \\ &= ss' + s\eta(b_1) + s'\eta(a_1) + \eta(a_1)\eta(b_1) + s\eta(b_2) + s'\eta(a_2) + \eta(a_1 b_2 + a_2(b_1 + b_2)) \\ & \quad \dots + s\eta(b_k) + s'\eta(a_k) + \sum_{i=1}^{k-1} \eta(a_i)\eta(b_k) + \eta(a_k) \sum_{i=1}^k \eta(b_i) \\ &= s(s' + \sum_{i=1}^k \eta(b_i)) + (\sum_{i=1}^k \eta(a_i))(s' + \sum_{i=1}^k \eta(b_i)) \\ &= (s + \eta(a_1 + \dots + a_k))(s' + \eta(b_1 + \dots + b_k)) \\ &= \eta_k(s, a_1, \dots, a_k) \eta_k(s', b_1, \dots, b_k). \end{aligned}$$

(ii) Let

$$u = (s, a_1, \dots, a_k), v = (s', b_1, \dots, b_k) \in B_k.$$

Then we obtain

$$\begin{aligned} d_0(u * v) &= d_0(ss', s \cdot b_1 + s' \cdot a_1 + a_1 b_1, s \cdot b_2 + s' \cdot a_2 + a_1 b_2 + a_2(b_1 + b_2), \\ &\quad \dots, s \cdot b_k + s' \cdot a_k + \sum_{i=1}^{k-1} a_i b_k + a_k \sum_{i=1}^k b_i) \\ &= (ss' + s\eta(b_1) + s'\eta(a_1) + \eta(a_1)\eta(b_1), s \cdot b_2 + s' \cdot a_2 + a_1 b_2 + a_2(b_1 + b_2), \\ &\quad \dots, s \cdot b_k + s' \cdot a_k + \sum_{i=1}^{k-1} a_i b_k + a_k \sum_{i=1}^k b_i) \\ &= ((s + \eta(a_1)(s' + \eta(b_1))), s \cdot b_2 + s' \cdot a_2 + a_1 b_2 + a_2(b_1 + b_2), \\ &\quad \dots, s \cdot b_k + s' \cdot a_k + \sum_{i=1}^{k-1} a_i b_k + a_k \sum_{i=1}^k b_i) \\ &= (s + \eta a_1, a_2, \dots, a_k)(s' + \eta b_1, b_2, \dots, b_k) \\ &= d_0(u) * d_0(v). \end{aligned}$$

(iii) Let

$$u = (s, a_1, \dots, a_k), v = (s', b_1, \dots, b_k) \in B_k.$$

We shall show that the k -module homomorphisms d_i are k -algebra homomorphisms for $0 \leq$

$i \leq k - 1$. We calculate

$$\begin{aligned}
d_i(u * v) &= d_i(ss', s \cdot b_1 + s' \cdot a_1 + a_1 b_1, s \cdot b_2 + s' \cdot a_2 + a_1 b_2 + a_2(b_1 + b_2), \\
&\quad \dots, s \cdot b_k + s' \cdot a_k + \sum_{i=1}^{k-1} a_i b_k + a_k \sum_{i=1}^k b_i) \\
&= (ss', s \cdot b_1 + s' \cdot a_1 + a_1 b_1, s \cdot b_2 + s' \cdot a_2 + a_1 b_2 + a_2(b_1 + b_2), \\
&\quad \dots, s \cdot b_{i-1} + s' \cdot a_{i-1} + b_{i-1} \sum_{j=1}^{i-2} a_j + a_{i-1} \sum_{j=1}^{i-1} b_j \\
&\quad + s \cdot b_i + s' \cdot a_i + b_i \sum_{j=1}^{i-1} a_j + a_i \sum_{j=1}^i b_j, \\
&\quad \dots, s \cdot b_k + s' \cdot a_k + \sum_{i=1}^{k-1} a_i b_k + a_k \sum_{i=1}^k b_i) \\
&= (ss', s \cdot b_1 + s' \cdot a_1 + a_1 b_1, \dots, s' \cdot (a_{i-1} + a_i) + s \cdot (b_{i-1} + b_i) \\
&\quad + (b_{i-1} + b_i) \sum_{j=1}^{i-2} a_j + (a_{i-1} + a_i) \sum_{j=1}^{i-1} b_j + (a_{i-1} + a_i) b_i, \\
&\quad \dots, s \cdot b_k + s' \cdot a_k + \sum_{i=1}^{k-1} a_i b_k + a_k \sum_{i=1}^k b_i) \\
&= (s, a_1, \dots, a_{i-1} + a_i, \dots, a_k)(s', b_1, \dots, b_{i-1} + b_i, \dots, b_k) \\
&= d_i(u) * d_i(v)
\end{aligned}$$

for $0 \leq i \leq k - 1$, so Part (iii) holds.

(iv) Since in any semi-direct product, projection onto the first coordinate is a homomorphism, the projection map

$$d_k : (s, a_1, \dots, a_k) \mapsto (s, a_1, \dots, a_{k-1})$$

is a homomorphism from B_k to B_{k-1} for $k \geq 1$.

(v) We leave it to the reader. \square

4 The mutual inverse relation between above associations

Let $\eta : R \rightarrow S$ be an algebra homomorphism. We showed that how to start with an ideal simplicial algebra structure on $Bar(S, R)$ and obtain a crossed module structure $l : S \rightarrow End(R)$ on η and we showed how to start with a crossed module structure on η and obtain an ideal simplicial algebra structure on the simplicial k -module $Bar(S, R)$. Our aim in this section is to make the observation that these two associations are mutual inverses.

Assume first that the simplicial k -module $Bar(S, R)$ is endowed with an ideal simplicial algebra structure, and denote the multiplication in B_k as

$$(s, a_1, \dots, a_k) * (s', b_1, \dots, b_k).$$

We showed that the action $l : S \rightarrow End(R)$ given by $l_s : r \mapsto s \cdot r$ gives an crossed module structure on η . Further given this crossed module structure on η the equation

$$(s, a_1, \dots, a_k) * (s', b_1, \dots, b_k) = (ss', s \cdot b_1 + s' \cdot a_1 + a_1 b_1, s \cdot b_2 + s' \cdot a_2 + a_1 b_2 + a_2(b_1 + b_2), \\ \dots, s \cdot b_k + s' \cdot a_k + \sum_{i=1}^{k-1} a_i b_k + a_k \sum_{i=1}^k b_i).$$

given above tells us how to define an ideal simplicial algebra structure on B_k with the multiplication ‘*’.

Conversely let $l : S \rightarrow End(R)$ be an ideal structure (or a crossed module structure) on η . Let ‘*’ be the multiplication in B_k as given above. Let $l' : S \rightarrow End(R)$ be the crossed module structure on η . That is for all $s \in S$, $l'_s : r \mapsto s' \cdot r$ where $(0, s') = (s, 0) * (0, r)$. Now by definition of the operation *, we obtain

$$(s, 0) * (0, r) = (0s, s \cdot r + 0 \cdot 0 + 0 \cdot r) = (0, s \cdot r).$$

We thus see that $s' = s \cdot r$ for all $r \in R$, $s \in S$, that is $l'_s = l_s$ for all $s \in S$. This completes the observation that the two associations are mutual inverses.

5 Crossed ideal maps between ideal maps

As it is well known and explored above that an ideal structure over a homomorphism of algebras $\eta : R \rightarrow S$ preserves the ideals. So we can say that if there is an ideal structure over η , then $\eta(R)$ is an ideal of S . This ideal approach to crossed modules shades some light on Loday’s crossed square (cf. [13]). In this section we will give an extension of this result for higher dimensional crossed modules. We will prove that if there is a (crossed) ideal structure over a morphism between crossed modules, then this map preserves the (crossed) ideals. First we give the notion of ‘crossed ideal’ of a crossed module of algebras from [14].

Definition 5.1 *A homomorphism of algebras $\eta' : R' \rightarrow S'$ will be called a crossed ideal of the crossed module $\eta : R \rightarrow S$ in the category of crossed modules over k -algebras if:*

$\mathfrak{CJ1} : \eta' : R' \rightarrow S'$ is a subcrossed module of $\eta : R \rightarrow S$, that is, the following conditions are satisfied:

- (i) *R' is a subalgebra of R and S' is a subalgebra of S .*
- (ii) *the action of S' on R' induced by the action of S on R .*
- (iii) *$\eta' : R' \rightarrow S'$ is a crossed module.*

(iv) the following diagram of morphisms of crossed modules

$$\begin{array}{ccc} R' & \xrightarrow{\mu} & R \\ \eta' \downarrow & & \downarrow \eta \\ S' & \xrightarrow{\nu} & S \end{array}$$

commutes, where μ and ν are the inclusions,

$$\mathfrak{CJ2} : R'R = RR' \subseteq R' \text{ and } SS' = S'S \subseteq S',$$

$$\mathfrak{CJ3} : R \cdot S' = S' \cdot R \subseteq R',$$

$$\mathfrak{CJ4} : R' \text{ is closed under the action of } S, \text{ i.e. } S \cdot R' = R' \cdot S \subseteq R'.$$

5.1 Crossed ideal structure over maps between crossed modules

Assume that $\eta_1 : R_1 \rightarrow S_1$ and $\eta_2 : R_2 \rightarrow S_2$ are crossed modules. Let $\alpha : (\alpha_1, \alpha_2)$ be a morphism from η_1 to η_2 in the category of crossed modules of k -algebras, where $\alpha_1 : R_1 \rightarrow R_2$ and $\alpha_2 : S_1 \rightarrow S_2$ are homomorphisms of k -algebras. In this case, the morphism $\alpha := (\alpha_1, \alpha_2)$ satisfies the following conditions:

(i) the diagram

$$\begin{array}{ccc} R_1 & \xrightarrow{\alpha_1} & R_2 \\ \eta_1 \downarrow & & \downarrow \eta_2 \\ S_1 & \xrightarrow{\alpha_2} & S_2 \end{array}$$

commutes, i.e. $\alpha_2\eta_1 = \eta_2\alpha_1$,

(ii) for all $s_1 \in S_1$ and $r_1 \in R_1$,

$$\alpha_1(l_{s_1}(r_1)) = l_{\alpha_2(s_1)}(\alpha_1(r_1)).$$

Definition 5.2 A morphism $\alpha := (\alpha_1, \alpha_2)$ between crossed modules η_1 and η_2 is called a ‘crossed ideal map’ if

(i) there are ideal map structures over the homomorphisms α_1, α_2 and $\eta_2\alpha_1 = \alpha_2\eta_1$, and
(ii) there is an S_2 -bilinear map $h : R_2 \times S_1 \rightarrow R_1$ satisfying the conditions:

$$(a) \alpha_1(h(r_2, s_1)) = l_{\alpha_2(s_1)}(r_2),$$

$$(b) \eta_1(h(r_2, s_1)) = l_{\eta_2(r_2)}(s_1),$$

$$(c) h(\alpha_1(r_1), s_1) = l_{s_1}(r_1),$$

$$(d) h(r_2, \eta_1(r_1)) = l_{r_2}(r_1).$$

for all $r_2 \in R_2, s_1 \in S_1$.

Remark 5.3 In fact, a crossed ideal structure over the map α , between crossed modules η_1 and η_2 , gives a crossed square structure of algebras on the square

$$\begin{array}{ccc} R_1 & \xrightarrow{\alpha_1} & R_2 \\ \eta_1 \downarrow & & \downarrow \eta_2 \\ S_1 & \xrightarrow{\alpha_2} & S_2 \end{array}$$

defined by Ellis [6] and introduced by Guin-Waléry and Loday, [13], in the group case.

Thus we get the following result.

Proposition 5.4 If the morphism $\alpha : (\alpha_1, \alpha_2)$ is a crossed ideal map from $\eta_1 : R_1 \rightarrow S_1$ to $\eta_2 : R_2 \rightarrow S_2$ in the category of crossed modules of k -algebras, then $\alpha(\eta_1) : \alpha_1(R_1) \rightarrow \alpha_2(S_1)$ is a crossed ideal of the crossed module $\eta_2 : R_2 \rightarrow S_2$.

Proof: First, we consider the following square

$$\begin{array}{ccc} \alpha_1(R_1) = R'_1 & \xrightarrow{\mu} & R_2 \\ \overline{\eta_2} \downarrow & & \downarrow \eta_2 \\ \alpha_2(S_1) = S'_1 & \xrightarrow[\nu]{} & S_2 \end{array}$$

where μ and ν are the inclusions. The map $\overline{\eta_2} : R'_1 \rightarrow S'_1$ is defined by the restriction of the map η_2 to the subalgebra $\alpha_1(R_1)$ of R_2 . We will show that the restricted homomorphism $\overline{\eta_2}$ is a crossed ideal of η_2 .

℄1. First we will show that $\overline{\eta_2}$ is a subcrossed module of η_2 .

(i) It is clear that R'_1 is a subalgebra of R_2 and similarly $\alpha_2(S_1) = S'_1$ is a subalgebra of S_2 .

(ii) Since the map $\alpha := (\alpha_1, \alpha_2)$ is a crossed module morphism, it satisfies the condition $l_{\alpha_2(s_1)}(\alpha_1(r_1)) = \alpha_1(l_{s_1}(r_1))$ for all $r_1 \in R_1$ and $s_1 \in S_1$. Then the algebra action of $\alpha_2(s_1) \in S'_1$ on $\alpha_1(r_1) \in R'_1$ can be given by $\alpha_2(s_1) \cdot \alpha_1(r_1) = \alpha_1(s_1 \cdot r_1) \in R'_1$.

(iii) We will show that $\overline{\eta_2} : R'_1 \rightarrow S'_1$ is a crossed module. For all $\alpha_2(s_1) \in S'_1$ and $\alpha_1(r_1), \alpha_1(r'_1) \in R'_1$, we obtain

$$\begin{aligned} \overline{\eta_2}(l_{\alpha_2(s_1)}(\alpha_1(r_1))) &= \eta_2 \alpha_1(l_{s_1}(r_1)) \\ &= \alpha_2 \eta_1(l_{s_1}(r_1)) \\ &= \alpha_2(s_1 \eta_1(r_1)) \quad (\text{since } \eta_1 \text{ is a crossed module}) \\ &= \alpha_2(s_1) \alpha_2 \eta_1(r_1) \\ &= \alpha_2(s_1) \eta_2 \alpha_1(r_1) \\ &= \alpha_2(s_1) \overline{\eta_2}(\alpha_1(r_1)), \end{aligned}$$

and

$$\begin{aligned} l_{\overline{\eta_2}(\alpha_1(r_1))} \alpha_1(r'_1) &= l_{\alpha_2(\eta_1(r_1))} \alpha_1(r'_1) \\ &= \alpha_1(l_{\eta_1(r_1)}(r'_1)) \\ &= \alpha_1(r_1 r'_1) \quad (\text{since } \eta_1 \text{ is a crossed module}) \\ &= \alpha_1(r_1) \alpha_1(r'_1). \end{aligned}$$

(iv) the square

$$\begin{array}{ccc} R'_1 & \xrightarrow{\mu} & R_2 \\ \overline{\eta_2} \downarrow & & \downarrow \eta_2 \\ S'_1 & \xrightarrow{\nu} & S_2 \end{array}$$

is commutative, because μ and ν are the inclusions and $\overline{\eta_2}$ is given by the restriction of η_2 . Thus $\overline{\eta_2}$ is a subcrossed module of η_2 .

3.2. Since there are ideal structures over the maps $\alpha_1 : R_1 \rightarrow R_2$ and $\alpha_2 : S_1 \rightarrow S_2$, we obtain that $\alpha_1(R_1) = R'_1$ and $\alpha_2(S_1) = S'_1$ are ideals of R_2 and S_2 respectively. Therefore, we obtain

$$R'_1 R_2 = R_2 R'_1 \subseteq R'_1 \text{ and } S'_1 S_2 = S_2 S'_1 \subseteq S'_1.$$

3.3. We have to show that $R_2 \cdot S'_1 = S'_1 \cdot R_2 \subseteq R'_1$. We use the h -map to prove it. For all $\alpha_2(s_1) \in S'_1$ and $r_2 \in R_2$ we have $r_2 \cdot \alpha_2(s_1) = \alpha_2(s_1) \cdot r_2 = l_{\alpha_2(s_1)}(r_2) = \alpha_1(h(r_2, s_1))$, where $h(r_2, s_1) \in R_1$, then we obtain $r_2 \cdot \alpha_2(s_1) = \alpha_2(s_1) \cdot r_2 \in \alpha_1(R_1) = R'_1$ so that $R_2 \cdot S'_1 = S'_1 \cdot R_2 \subseteq R'_1$.

3.4. We have to show that $S_2 \cdot R'_1 = R'_1 \cdot S_2 \subseteq R'_1$. For all $s_2 \in S_2$ and $\alpha_1(r_1) \in R'_1$, we can define the action by $s_2 \cdot \alpha_1(r_1) = l_{s_2}(\alpha_1(r_1)) = \alpha_1(l_{s_1}(r_1)) \in R'_1$. Thus R'_1 is closed under the action of S_2 and this completes the proof. \square

Conversely, as it was illustrated in [14], we can easily say that given a crossed ideal $\overline{\eta_2} : R'_1 \rightarrow S'_1$ of the crossed module $\eta_2 : R_2 \rightarrow S_2$, then inclusion morphism from $\overline{\eta_2}$ to η_2 is a crossed ideal map in the category of crossed modules of k -algebras.

Indeed, in the following diagram

$$\begin{array}{ccc} R'_1 & \xrightarrow{\mu} & R_2 \\ \overline{\eta_2} \downarrow & & \downarrow \eta_2 \\ S'_1 & \xrightarrow{\nu} & S_2 \end{array}$$

if $\overline{\eta_2}$ is a crossed ideal of η_2 , the inclusion morphisms μ and ν become crossed modules with the natural actions of R_2 and S_2 on their ideals R'_1 and S'_1 given by the multiplication, respectively. Further, the h -map $h : R_2 \times S'_1 \rightarrow R'_1$ is defined by $h(r_2, s'_1) = (l|_{S'_1})_{s'_1}(r_2)$, where $l|_{S'_1}$ is the restriction of $l : S_2 \rightarrow \text{End}(R_2)$ to S'_1 .

6 From the morphism $\alpha : \eta_1 \rightarrow \eta_2$ to the usual Bar construction

As mentioned above, in [8], Farjoun and Segev proved that a crossed module map $l : G \rightarrow \text{Aut}(N)$, which they call a normal structure on the map $N \rightarrow G$ is inversely associated with a group structure on the homotopy quotient $G//N := \text{hocolim}_N G$ by taking $G//N$ to be the usual Bar construction. They also stated in section 6 of their work, for a morphism from a normal map $X \rightarrow G$ to a normal map $Y \rightarrow H$ in the category of normal maps, one can form a

simplicial group morphism $X//G \rightarrow Y//H$ and the homotopy quotient $(Y//H)/(X//G)$. In fact, if there is a normal map structure over the simplicial group morphism $X//G \rightarrow Y//H$, then $(Y//H)/(X//G)$ becomes a *bisimplicial* group, algebraically. In this section, we make some remarks concerning these ideas over k -algebras.

Recall that a functor $\mathbf{E}_{.,.} : (\Delta \times \Delta)^{op} \rightarrow \mathbf{Alg}$ is called a bisimplicial algebra, where Δ is the category of finite ordinals and \mathbf{Alg} is the category of (commutative) k -algebras. Hence $\mathbf{E}_{.,.}$ is equivalent to giving for each (p, q) an algebra $E_{p,q}$ and morphisms:

$$\begin{aligned} d_i^{h(pq)} &: E_{p,q} \rightarrow E_{p-1,q} \\ s_i^{h(pq)} &: E_{p,q} \rightarrow E_{p+1,q} \quad 0 \leq i \leq p \\ d_j^{v(pq)} &: E_{p,q} \rightarrow E_{p,q-1} \\ s_j^{v(pq)} &: E_{p,q} \rightarrow E_{p,q+1} \quad 0 \leq j \leq q \end{aligned}$$

such that the maps $d_i^{h(pq)}, s_i^{h(pq)}$ commute with $d_j^{v(pq)}, s_j^{v(pq)}$ and that $d_i^{h(pq)}, s_i^{h(pq)}$ (resp. $d_j^{v(pq)}, s_j^{v(pq)}$) satisfy the usual simplicial identities.

Now we suppose that $\alpha : (\alpha_1, \alpha_2)$ is a morphism from $\eta_1 : R_1 \rightarrow S_1$ to $\eta_2 : R_2 \rightarrow S_2$ in the category of crossed modules of k -algebras. Using the usual Bar construction, we can form the simplicial algebras $S_1//R_1$ and $S_2//R_2$ from η_1 and η_2 respectively as above. Analogously to [8], we obtain a simplicial algebra morphism

$$\Phi : S_1//R_1 \rightarrow S_2//R_2$$

and we can define this map on each step by

$$\Phi_n : (S_1 \ltimes (R_1)^{\ltimes n}) \rightarrow (S_2 \ltimes (R_2)^{\ltimes n})$$

with

$$\Phi_n : (s_1, r_1, r_2, \dots, r_n) = (\alpha_2(s_1), \alpha_1(r_1), \alpha_1(r_2), \dots, \alpha_1(r_n))$$

for all $s_1 \in S_1$ and $r_i \in R_1$ and where the maps Φ_n are homomorphisms of algebras.

An action of the algebra $(S_1 \ltimes (R_1)^{\ltimes n})$ on the underlying k -module of the algebra $(S_2 \ltimes (R_2)^{\ltimes n})$ can be given by this map, namely,

$$(s_1, \ltimes_{i=1}^n(r_i)) : (s_2, \ltimes_{i=1}^n(r'_i)) = (s_2 + \alpha_1(s_1), \ltimes_{i=1}^n(r'_i + \alpha_1(r_i)))$$

where $s_1 \in S_1, s_2 \in S_1$ and $r_i \in R_1, r'_i \in R_2$ for $i = 1, 2, \dots, n$.

Using this action, on each step and considering the usual Bar construction, we can form a *bisimplicial* k -module,

$$\mathfrak{Bar}^{(2)} : (S_2//R_2)/(S_1//R_1)$$

and, on each directions, this can be defined by the k -modules

$$\mathfrak{Bar}_{n,m}^{(2)} := (S_2 \ltimes (R_2)^{\ltimes n}) \times (S_1 \ltimes (R_1)^{\ltimes n})^{\times m}.$$

The horizontal homomorphisms between these k -modules can be defined as follows:

1. For all

$$(s_2, r_{21}, \dots, r_{2n}) \in S_2 \times (R_2)^{\times n}$$

and

$$((s_1^1, r_{11}^1, \dots, r_{1n}^1), \dots, (s_1^m, r_{11}^m, \dots, r_{1n}^m)) \in (S_1 \times (R_1)^{\times n})^{\times m},$$

where, for $1 \leq i \leq n$ and $1 \leq j \leq m$, $r_{1i}^j \in R_1$, $r_{2i} \in R_2$, $s_2 \in S_2$, $s_1^j \in S_1$, the $d_0^h : \mathfrak{Bar}_{n,m}^{(2)} \rightarrow \mathfrak{Bar}_{n,m-1}^{(2)}$ is defined by

$$\begin{aligned} d_0^h((s_2, r_{21}, \dots, r_{2n}), (s_1^1, r_{11}^1, \dots, r_{1n}^1), \dots, (s_1^m, r_{11}^m, \dots, r_{1n}^m)) \\ = ((s_2, r_{21}, \dots, r_{2n}) + \Phi_n(s_1^1, r_{11}^1, \dots, r_{1n}^1), (s_1^2, r_{11}^2, \dots, r_{1n}^2), \dots, (s_1^m, r_{11}^m, \dots, r_{1n}^m)). \end{aligned}$$

2. For $0 < i < m$, the $d_i^h : \mathfrak{Bar}_{n,m}^{(2)} \rightarrow \mathfrak{Bar}_{n,m-1}^{(2)}$ is defined by

$$\begin{aligned} d_i^h((s_2, r_{21}, \dots, r_{2n}), (s_1^1, r_{11}^1, \dots, r_{1n}^1), \dots, (s_1^m, r_{11}^m, \dots, r_{1n}^m)) \\ = ((s_2, r_{21}, \dots, r_{2n}), (s_1^1, r_{11}^1, \dots, r_{1n}^1), \dots, \\ (s_1^i, r_{11}^i, \dots, r_{1n}^i) + (s_1^{i+1}, r_{11}^{i+1}, \dots, r_{1n}^{i+1}), \dots, (s_1^m, r_{11}^m, \dots, r_{1n}^m)). \end{aligned}$$

3. $d_m^h : \mathfrak{Bar}_{n,m}^{(2)} \rightarrow \mathfrak{Bar}_{n,m-1}^{(2)}$ is defined by

$$\begin{aligned} d_m^h((s_2, r_{21}, \dots, r_{2n}), (s_1^1, r_{11}^1, \dots, r_{1n}^1), \dots, (s_1^m, r_{11}^m, \dots, r_{1n}^m)) \\ = ((s_2, r_{21}, \dots, r_{2n}), (s_1^1, r_{11}^1, \dots, r_{1n}^1), \dots, (s_1^{m-1}, r_{11}^{m-1}, \dots, r_{1n}^{m-1})). \end{aligned}$$

4. For all $0 \leq i \leq m$, the $s_i^h : \mathfrak{Bar}_{n,m}^{(2)} \rightarrow \mathfrak{Bar}_{n,m+1}^{(2)}$ is defined by

$$\begin{aligned} s_i^h((s_2, r_{21}, \dots, r_{2n}), (s_1^1, r_{11}^1, \dots, r_{1n}^1), \dots, (s_1^m, r_{11}^m, \dots, r_{1n}^m)) \\ = ((s_2, r_{21}, \dots, r_{2n}), (s_1^1, r_{11}^1, \dots, r_{1n}^1), \dots, \\ (s_1^i, r_{11}^i, \dots, r_{1n}^i), (0, 0, \dots, 0), (s_1^{i+1}, r_{11}^{i+1}, \dots, r_{1n}^{i+1}), \dots, (s_1^m, r_{11}^m, \dots, r_{1n}^m)). \end{aligned}$$

Similarly, the vertical homomorphisms can be defined as follows:

1. the $d_0^v : \mathfrak{Bar}_{n,m}^{(2)} \rightarrow \mathfrak{Bar}_{n-1,m}^{(2)}$ is defined by

$$\begin{aligned} d_0^v((s_2, r_{21}, \dots, r_{2n}), (s_1^1, r_{11}^1, \dots, r_{1n}^1), \dots, (s_1^m, r_{11}^m, \dots, r_{1n}^m)) \\ = ((s_2 + \eta_2(r_{21}), r_{22}, \dots, r_{2n}), (s_1^1 + \eta_1(r_{11}^1), r_{12}^2, \dots, r_{1n}^2), \dots, (s_1^m + \eta_1(r_{11}^m), r_{12}^m, \dots, r_{1n}^m)). \end{aligned}$$

2. For $0 < i < n$, the $d_i^v : \mathfrak{Bar}_{n,m}^{(2)} \rightarrow \mathfrak{Bar}_{n-1,m}^{(2)}$ is defined by

$$\begin{aligned} d_i^v((s_2, r_{21}, \dots, r_{2n}), (s_1^1, r_{11}^1, \dots, r_{1n}^1), \dots, (s_1^m, r_{11}^m, \dots, r_{1n}^m)) \\ = ((s_2, r_{21}, \dots, r_{2i} + r_{2(i+1)}, \dots, r_{2n}), (s_1^1, r_{11}^1, \dots, r_{1i}^1 + r_{1(i+1)}^1, \dots, r_{1n}^1), \dots, \\ (s_1^m, r_{11}^m, \dots, r_{1i}^m + r_{1(i+1)}^m, \dots, r_{1n}^m)). \end{aligned}$$

3. $d_n^v : \mathfrak{Bar}_{n,m}^{(2)} \rightarrow \mathfrak{Bar}_{n-1,m}^{(2)}$ is defined by

$$\begin{aligned} d_n^v((s_2, r_{21}, \dots, r_{2n}), (s_1^1, r_{11}^1, \dots, r_{1n}^1), \dots, (s_1^m, r_{11}^m, \dots, r_{1n}^m)) \\ = ((s_2, r_{21}, \dots, r_{2(n-1)}), (s_1^1, r_{11}^1, \dots, r_{1(n-1)}^1), \dots, (s_1^{m-1}, r_{11}^{m-1}, \dots, r_{1(n-1)}^{m-1})). \end{aligned}$$

4. For all $0 \leq i \leq n$, the $s_i^v : \mathfrak{Bar}_{n,m}^{(2)} \rightarrow \mathfrak{Bar}_{n+1,m}^{(2)}$ is defined by

$$\begin{aligned} s_i^v((s_2, r_{21}, \dots, r_{2n}), (s_1^1, r_{11}^1, \dots, r_{1n}^1), \dots, (s_1^m, r_{11}^m, \dots, r_{1n}^m)) \\ = ((s_2, r_{21}, \dots, r_{2i}, 0, r_{2(i+1)}, \dots, r_{2n}), (s_1^1, r_{11}^1, \dots, r_{1i}^1, 0, r_{1(i+1)}^1, \dots, r_{1n}^1), \dots, \\ (s_1^m, r_{11}^m, \dots, r_{1i}^m, 0, r_{1(i+1)}^m, \dots, r_{1n}^m)). \end{aligned}$$

In low dimensions, we can illustrate this bisimplicial k -module by the diagram:

$$\begin{array}{ccccc} \dots & \xRightarrow{\quad} & (S_2 \ltimes (R_2)^2) \times (S_1 \ltimes (R_1)^2) & \xRightarrow{\quad} & (S_2 \ltimes R_2^2) \\ \downarrow \downarrow \downarrow & & \downarrow \downarrow & & \downarrow \downarrow \\ (S_2 \ltimes R_2) \times (S_1 \ltimes R_1)^2 & \xRightarrow{\quad} & (S_2 \ltimes R_2) \times (S_1 \ltimes R_1) & \xRightarrow{\quad} & S_2 \ltimes R_2 \\ \downarrow \downarrow & & \downarrow \downarrow & & \downarrow \downarrow \\ \dots S_2 \times (S_1)^2 & \xRightarrow{\quad} & (S_2 \times S_1) & \xRightarrow{\quad} & S_2 \end{array}$$

For instance, in this diagram, the homomorphisms in the first square are given by:

$$\begin{aligned} d_0^v(s_2, r_2) &= s_2 + \eta_2 r_2, & d_0^h(s_2, s_1) &= s_2 + \alpha_2(s_1) \\ d_1^v(s_2, r_2) &= s_2, & d_1^h(s_2, s_1) &= s_2 \\ s_0^v(s_2) &= (s_2, 0), & s_0^h(s_2) &= (s_2, 0). \end{aligned}$$

and

$$\begin{aligned} d_0^v(s_2, r_2, s_1, r_1) &= (s_2 + \eta_2 r_2, s_1 + \eta_1 r_1), & d_0^h(s_2, r_2, s_1, r_1) &= (s_2 + \alpha_2(s_1), r_2 + \alpha_1(r_1)) \\ d_1^v(s_2, r_2, s_1, r_1) &= (s_2, s_1), & d_1^h(s_2, r_2, s_1, r_1) &= (s_2, r_2) \\ s_0^v(s_2, s_1) &= (s_2, 0, s_1, 0), & s_0^h(s_2, r_2) &= (s_2, r_2, 0, 0). \end{aligned}$$

Therefore, we obtained a bisimplicial k -module, from the morphism α in the category of crossed modules of k -algebras. Thus we get the following result.

Theorem 6.1 Given a morphism $\alpha : \eta_1 \rightarrow \eta_2$ in the category of crossed modules of k -algebras, a crossed ideal map structure on the morphism α gives an ideal bisimplicial algebra structure on the bisimplicial k -module $\mathfrak{Bar}^{(2)} : (S_2 // R_2) // (S_1 // R_1)$, and conversely, any ideal bisimplicial algebra structure on the bisimplicial k -module $\mathfrak{Bar}^{(2)} : (S_2 // R_2) // (S_1 // R_1)$ determines a crossed ideal map structure on the morphism $\alpha : \eta_1 \rightarrow \eta_2$.

Remark 6.2 Obviously, proving this theorem would take too much work and time. In order to prove it, we would need to give the notion of ‘ideal bisimplicial algebra structure’ over the

bisimplicial k -module $\mathfrak{Bar}^{(2)}$ explicitly. So we will clarify it in another work. Of course, this result can be iterated to the crossed n -cube structure defined by Ellis in [6]. In this case, we would need to give a detailed definition of a *crossed n -ideal* of a crossed n -cube and a *crossed n -ideal structure* over the morphism between crossed $(n - 1)$ cubes. Then it would give a multi-simplicial algebra in dimension n , or an n -simplicial algebra together with this structure.

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